Homework 1

Due: Thursday, October 5, 2023, 1pm on Gradescope

Please upload your answers timely to Gradescope. Start a new page for every problem. For the programming/simulation questions you can use any reasonable programming language. Comment your source code and include the code and a brief overall explanation with your answers.

1. **Exercise 1.1 in textbook.** Let A_1 and A_2 be arbitrary events and show that

$$\Pr\{A_1 \cup A_2\} + \Pr\{A_1 \cap A_2\} = \Pr\{A_1\} + \Pr\{A_2\}$$

   Explain which parts of the sample space are being double counted on both sides of this equation and which parts are being counted once.

2. **Exercise 1.12 in textbook, parts (a), (b).** Let X be a random variable with CDF $F(x)$. Find the CDF of the following random variables.
   (a) The maximum of n i.i.d. rvs, each with CDF $F(x)$.
   (b) The minimum of n i.i.d. rvs, each with CDF $F(x)$.

3. **Exercise 1.21 in textbook, parts (a)-(e).**
   (a) Show that, for uncorrelated rvs, the expected value of the product is equal to the product of the expected values (by definition, X and Y are uncorrelated if $\mathbb{E}[(X - \bar{X})(Y - \bar{Y})] = 0$).
   (b) Show that if X and Y are uncorrelated, then the variance of $X + Y$ is equal to the variance of X plus the variance of Y.
   (c) Show that if $X_1 \ldots, X_n$ are uncorrelated, then the variance of the sum is equal to the sum of the variances.
   (d) Show that independent rv s are uncorrelated.
   (e) Let X, Y be identically distributed ternary valued rvs with the PMF $p_X(-1) = p_X(1) = 1/4; p_X(0) = 1/2$. Find a simple joint probability assignment such that X and Y are uncorrelated but dependent.

4. Consider the binary symmetric channel model mentioned in class. This is a channel we can use to transmit bits and each bit is flipped independently with probability p regardless of the value of the bit. We want to estimate the flip probability p of the channel by sending a known training bit sequence over the channel and counting the number of bit flips at the receiver side.

a) Suppose $p = 0.1$ and we want to estimate it to within accuracy plus or minus 0.01. We want to know the minimum length n^* of the training sequence (i.e. the number of bits you need to send over the channel) such that:

$$\Pr\{|\hat{p_n} - p| > 0.01\} < 0.05,$$

where $\hat{p_n}$ is the fraction of bits flipped. Do this in two ways and compare your answers:

i. Bound using Chebyshev bound

ii. Estimate using the Central-limit approximation

b) In practice p is unknown. (That's the whole point!) Repeat part (a) to come up with sensible answers using both ways. You may do the calculations either analytically or using a short computer program.

Hint: Can you identify the worst-case value of p that will make the bounds you derived in part (a) largest?

5. **Two Envelopes**
A fixed amount a is placed in one envelope and an amount $5a$ is placed in the other. Suppose one of the two envelopes is chosen uniformly at random, and let X be the amount in the opened envelope. Let Y be the amount in the other envelope.

a) Find $\mathbb{E}\left[\frac{Y}{X}\right]$.

b) Find $\mathbb{E}\left[\frac{X}{Y}\right]$.

c) Find $\mathbb{E}[Y]/\mathbb{E}[X]$.

6. **Computing probability of events**

(a) **Uniform r.v.** Consider a random variable $X \sim \text{Unif}[2, 10]$, i.e., uniformly chosen from the interval $[2, 10]$. What is the probability that $X^2 - 12X + 35 > 0$?

(b) **Laplacian r.v.** Let X be a continuous random variable with pdf $f_X(x) = (1/2)e^{-|x|}$ for $-\infty < x < \infty$. Find the probability of each of the following events.

i. $\{X \leq 2 \text{ or } X \geq 0\}$.

ii. $\{|X| + |X - 3| \leq 3\}$.